Ordered Reflection Systems in Cyclic Scales
Messiaen Modes, Multi-Octave Lifts, and Symmetric Voice Leading

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Abstract. Messiaen's modes of limited transposition are usually described as pitch-class sets whose transpositions repeat before all twelve chromatic levels are exhausted. This note keeps their ordered interval words and treats each mode as a finite reflection system. For each canonical mode, it enumerates modal rotations and reverses them, separating self-reflecting rotations from mirror pairs.

Mode 1 is self-reflecting; Modes 2 and 4 consist only of mirror pairs; and Modes 3, 5, 6, and 7 mix self-reflections with mirror pairs. This finite classification can be stated as a decidable theorem over cyclic words and checked in Lean. It complements set-theoretic accounts: limited transposition classifies recurrence of a pitch collection under transposition, while ordered reflection classifies how a path through that collection behaves under inversion. Multi-octave lifts then turn the finite reflection map into a registral and voice-leading resource. These uses remain structural and compositional, not perceptual claims.

Keywords. Messiaen; modes of limited transposition; interval words; cyclic words; reflection; Lean; voice leading.

I. Introduction

Messiaen's modes of limited transposition are pitch collections whose transpositions repeat before all twelve chromatic levels are exhausted [11]. In pitch-class terms, each collection has nontrivial rotational symmetry in $\mathbb{Z}_{12}$.

A mode also has an ordered cyclic interval structure. The interval word for Messiaen's third mode, for example, may be written as

$$2, 1, 1, 2, 1, 1, 2, 1, 1.$$

Rotating this word yields the distinct modal forms within the same collection. Reversing it reflects the cyclic order. The question is whether a rotation returns to itself or pairs with another rotation of the same interval word.

This is adjacent to pitch-class set theory and transformational theory [9, 10, 12, 14], but it tests a different object. Inversional symmetry of an unordered pitch-class set does not determine which ordered rotations are fixed by reversal.

The result is a word-level classification of reflection classes among Messiaen's modal rotations. Because the objects are finite tuples of positive integers, the table is also a compact formal-verification target: implement rotation, reversal, duplicate removal, and cyclic equivalence, then decide the seven cases by computation.

The larger claim is that the finite table is a seed for a compositional calculus. Limited transposition classifies when a pitch collection repeats under transposition; ordered reflection classifies how a path through that collection behaves under inversion; multi-octave lifting turns that finite reflection map into a registral and voice-leading resource.

The contribution is therefore deliberately modest but precise. It does not claim a new theory of Messiaen's harmonic practice, and it does not infer listener response from symmetry. It supplies a reproducible invariant for a familiar repertory of interval words, shows that the invariant is not reducible to unordered pitch-class inversion, records the full reversal maps, and indicates how those maps can drive controlled lifts, mirror alternations, and graph-like compositional moves.

II. Related Work

Messiaen's account of limited transposition appears in *The Technique of My Musical Language* [11]; Street gives an early English-language account of the modes as theoretical objects [15]. Cohn's study of transpositionally invariant sets gives the pitch-class baseline: a collection recurs under a nontrivial transposition before all twelve transpositions are exhausted [5]. Pitch-class set theory supplies the standard tools for unordered collections [9, 12, 14].

Here the object is an ordered cyclic interval word. Lewin's transformational theory supplies the broader relation-based setting [10]; Crans, Fiore, and Satyendra give the dihedral-action background [6]. Akhtar studies group actions on Messiaen's modes [1]. The distinction here is the focus on reversal of modal rotations.

Work on modes and scale words gives precedent for treating modal rotations as structured objects [3, 4]. Douthett and Steinbach place modes of limited transposition in parsimonious voice-leading graphs [8]; Neidhöfer develops a broader account of Messiaen's modal harmony and voice leading [13]. The present object is narrower: reflection of the mode's own interval word.

This narrower object is useful because it separates three questions that are often adjacent in prose: whether a pitch-class collection is invariant under transposition, whether it has inversional axes as an unordered set, and whether a particular ordered modal rotation is fixed by reversal. The first two questions concern the collection; the third concerns an ordered path through the collection.

III. Definitions

Let N be a chromatic modulus. The ordinary pitch-class case uses $N = 12$. A multi-octave lift uses $N = 12h$ for a positive integer height h.

Pitch-class set. A pitch-class set is a subset $S \subseteq \mathbb{Z}_N$. Transposition and inversion are

$$T_n(x) = x + n \pmod N, \qquad I_k(x) = k - x \pmod N.$$

Rotations and reflections generate the usual dihedral action on $\mathbb{Z}_N$.

Interval word. An ordered cyclic interval word is a tuple

$$w = (d_0, d_1, \dots, d_{m-1})$$

with positive integer steps summing to N:

$$\sum_{i=0}^{m-1} d_i = N.$$

The pitch path induced by w, starting at origin 0, is

$$P(w) = \{0, d_0, d_0 + d_1, \dots, d_0 + \cdots + d_{m-2}\} \pmod N.$$

The last interval closes the cycle and is not added as a new point.

Modal rotation. The rth modal rotation of w is

$$\mathsf{rot}_r(w) = (d_r, d_{r+1}, \dots, d_{r-1})$$

with indices modulo m. Let

$$\mathcal{M}(w) = \{\mathsf{rot}_r(w) : 0 \le r < m\}$$

with duplicate words identified.

Word reflection. The reflection of an interval word is its reversal:

$$\mathsf{rev}(w) = (d_{m-1}, d_{m-2}, \dots, d_0).$$

This reverses the cyclic order of intervals. In the Messiaen words below, the reflected word is then compared with the elements of $\mathcal{M}(w)$.

Self-reflection and mirror pairs. A modal rotation $u \in \mathcal{M}(w)$ is *self-reflecting* when $\mathsf{rev}(u)$ is cyclically equivalent to u. Two distinct rotations $u, v \in \mathcal{M}(w)$ form a *mirror pair* when $\mathsf{rev}(u)$ is cyclically equivalent to v and $u \neq v$.

Reflection system. The ordered reflection system of w is the finite directed structure

$$\mathcal{R}(w) = (\mathcal{M}(w), \mathsf{tgt}_w),$$

where tgt_w sends each modal rotation to the rotation class of its reversed word, when that class lies in $\mathcal{M}(w)$. A self-reflecting rotation is a fixed point of tgt_w; a mirror pair is a two-cycle. The fixedness ratio

$$F(w) = s(w)/|\mathcal{M}(w)|$$

measures only this marked ordered-reflection property. It is not a measure of acoustic consonance, perceptual stability, or total set-theoretic symmetry.

Set inversion versus word reflection. The pitch-class set induced by a rotation is $S(u) = P(u) \subseteq \mathbb{Z}_N$. It may be fixed by a pitch-class inversion even when the ordered interval word is not fixed by reversal. Set-level inversion and word-level reflection are therefore related but distinct tests.

Word reflection can be understood as the interval-word shadow of pitch inversion. Pitch inversion acts on points by $I_k(x) = k - x$; the induced traversal reverses the order of adjacent gaps. Thus a self-reflecting rotation preserves its ordered interval identity under this gap-level operation, while a mirror-pair rotation moves to a complementary modal identity.

IV. Computation

For each Messiaen word, enumerate its distinct rotations, reverse each rotation, and check whether the result is itself or another rotation.

A companion generator emits these Messiaen data, all positive interval words summing to 12, and selected lift datasets through $\mathbb{Z}_{48}$. A static companion page is generated from the same JSON files and is intended for publication at https://music.zacharyr0th.com/papers/messiaen-radial-symmetry. It renders staff notation, radial diagrams, the reflection matrix, and selected-word JSON from the dataset.

The seven input words are:

Mode	Interval word
M1	222222
M2	12121212
M3	211211211
M4	11311131
M5	141141
M6	22112211
M7	1112111121

Mode	Rot.	Map	Class
M1	1	$1 \to 1$	1 self
M2	2	$1 \leftrightarrow 2$	1 pair
M3	3	$1 \leftrightarrow 2$; $3 \to 3$	1 pair + 1 self
M4	4	$1 \leftrightarrow 2$; $3 \leftrightarrow 4$	2 pairs
M5	3	$1 \to 1$; $2 \leftrightarrow 3$	1 self + 1 pair
M6	4	$1 \leftrightarrow 3$; $2 \to 2$; $4 \to 4$	1 pair + 2 self
M7	5	$1 \leftrightarrow 3$; $2 \to 2$; $4 \leftrightarrow 5$	2 pairs + 1 self

Table 1: Ordered reflection classes of the seven canonical Messiaen interval words. Here "self" means self-reflecting rotation.

Mode	k	s	p	$F = s/k$
M1	1	1	0	1
M2	2	0	1	0
M3	3	1	1	1/3
M4	4	0	2	0
M5	3	1	1	1/3
M6	4	2	1	1/2
M7	5	1	2	1/5

Table 2: Ordered reflection fixedness for Messiaen's modes. Here $k = |\mathcal{M}(w)|$, s is the number of fixed rotations, and p is the number of unordered mirror pairs.

A. Formal theorem target

Let $\mathsf{tgt}_w(i)$ be the index of the modal rotation cyclically equivalent to $\mathsf{rev}(\mathsf{rot}_i(w))$, when that target lies in $\mathcal{M}(w)$. If no target lies in $\mathcal{M}(w)$, write $\mathsf{tgt}_w(i) = \bot$. Let

$$T(w) = (\mathsf{tgt}_w(1), \ldots, \mathsf{tgt}_w(k)),$$

where $k = |\mathcal{M}(w)|$. The ordered target map $T(w)$ is the invariant being classified in Tables 3 and 4. The count vector $C(w) = (s(w), p(w), e(w))$ is a useful summary, where s counts self-reflecting rotations, p counts unordered mirror pairs, and e counts reflections whose target lies outside $\mathcal{M}(w)$:

Mode	$C(M_i)$
M_1	$(1, 0, 0)$
M_2	$(0, 1, 0)$
M_3	$(1, 1, 0)$
M_4	$(0, 2, 0)$
M_5	$(1, 1, 0)$
M_6	$(2, 1, 0)$
M_7	$(1, 2, 0)$

The count vector is not by itself a complete ordered classifier: M3 and M5 both have $C = (1, 1, 0)$, but $T(M_3) = (2, 1, 3)$ while $T(M_5) = (1, 3, 2)$. The target map keeps that distinction. In a proof assistant, the obligation is small: define finite interval words, enumerate distinct rotations, reverse each word, quotient comparison by cyclic equivalence, and evaluate the resulting target map. The accompanying Lean package in `papers/messiaen-radial/formal` states the full table as `messiaen_reflection_classification`; the cases reduce directly from the definitions. The proof certifies the table from the definitions; it does not add a perceptual claim.

V. Results

Table 1 gives the reflection class of each canonical Messiaen interval word. In the map column, $1 \leftrightarrow 2$ marks a mirror pair, while $3 \to 3$ marks a self-reflecting rotation. Table 2 summarizes the same systems by rotation count, fixed points, mirror pairs, and fixedness ratio. Table 3 gives the complete modal-rotation maps, and Table 4 gives the same information as permutation matrices.

M3 is the smallest mixed case and a useful starting point for formalization. Its word

$$2, 1, 1, 2, 1, 1, 2, 1, 1$$

has three distinct rotations; reversal sends rotations 1 and 2 to one another and fixes rotation 3. Thus the local theorem for M3 already exercises both branches of the classifier.

Limited transposition and modal reflection are different invariants. M4 and M6 both have four distinct modal rotations; M4 has no fixed rotation, while M6 has two. In this narrow ordered sense, M6 has higher reflection fixedness than M4, even though both are strongly symmetric as pitch-class collections. Rotation count alone does not determine reflection class. Counts alone also do not determine ordered placement of fixed points: M3 and M5 each have one self-reflecting rotation and one mirror pair, but the self-reflecting rotation occurs at position 3 in M3 and at position 1 in M5.

Pitch-set axes versus modal fixed points. The unordered set answers "which pitches are present?" The ordered word answers "which path through them is being used?" A pitch-class set may have inversional axes in $\mathbb{Z}_{12}$ while its interval word belongs to a mirror pair rather than a fixed point. This is the distinction Table 1 isolates.

VI. Worked Example

The M3 word shows how the invariant can be used without adding a perceptual claim. Let

$$u_1 = 211211211, \quad u_2 = 112112112, \quad u_3 = 121121121$$

be the three rotations in Table 3. Reversal sends u_1 to the rotation class of u_2 and sends u_2 back to u_1, while u_3 is fixed. A composer or analyst can therefore distinguish a directed mirror pair from a balanced ordered form inside the same mode. The distinction is not audible by definition; it is a claim about the interval-word operation being applied.

Mode	Target vector	Full map by modal rotation
M1	(1)	1: 222222 → 1
M2	$(2, 1)$	1: 12121212 → 2; 2: 21212121 → 1
M3	$(2, 1, 3)$	1: 211211211 → 2; 2: 112112112 → 1; 3: 121121121 → 3
M4	$(2, 1, 4, 3)$	1: 11311131 → 2; 2: 13111311 → 1; 3: 31113111 → 4; 4: 11131113 → 3
M5	$(1, 3, 2)$	1: 141141 → 1; 2: 411411 → 3; 3: 114114 → 2
M6	$(3, 2, 1, 4)$	1: 22112211 → 3; 2: 21122112 → 2; 3: 11221122 → 1; 4: 12211221 → 4
M7	$(3, 2, 1, 5, 4)$	1: 1112111121 → 3; 2: 1121111211 → 2; 3: 1211112111 → 1; 4: 2111121111 → 5; 5: 1111211112 → 4

Table 3: Full ordered reflection maps for the seven canonical Messiaen words. The target vector lists $\mathtt{tgt}_w(i)$ for modal rotations in the displayed order.

Mode	Reflection matrix	Mode	Reflection matrix
M1	$\begin{bmatrix}1\end{bmatrix}$	M2	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$
M3	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	M4	$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$
M5	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	M6	$\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$
M7	$\begin{bmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix}$		

Table 4: Reflection matrices for all seven modes. Row i has a 1 in column $\mathtt{tgt}_w(i)$, so diagonal entries are self-reflecting rotations and symmetric off-diagonal entries are mirror pairs.

M5 gives the same count summary but a different ordered placement:

$$141141 \to 141141, \qquad 411411 \leftrightarrow 114114.$$

This is why the matrix or target vector is preferable to a count-only label when comparing modes. The count says how many fixed points and pairs exist; the full map says where they occur among the modal rotations.

VII. Transformation Graph

The reflection system can also be read as a small transformation graph. Each distinct modal rotation is a node. Rotation edges connect adjacent modal rotations; reflection edges connect u_i to $\mathtt{tgt}_w(i)$; lift edges connect a rotation in one octave or register to a chosen rotation in the next; voice-leading edges can be added with weights measuring a chosen cost, such as total registral motion.

This graph interpretation is not a new acoustic theory. It gives a composer or analyst a finite control surface. A self-reflecting node can serve as an inversional pivot because the reflection edge returns to the same ordered identity. A mirror pair supplies directed contrast because reflection changes the modal identity. A path from a mirror-pair node to a self-reflecting node can therefore be used as a structural resolution of reflectional asymmetry into fixed reflection. That is a formal cadence-like motion, not a tonal dominant–tonic cadence and not an automatic perceptual effect.

VIII. Diatonic Comparison

The same classifier gives a first asymmetry comparison. The major-diatonic interval word

$$2, 2, 1, 2, 2, 2, 1$$

has seven distinct modal rotations. As a pitch-class collection it has no nonzero transposition that fixes it in $\mathbb{Z}_{12}$, so it is not a mode of limited transposition in Messiaen's sense. Under the ordered reflection test, however, it is not structureless:

$$1 \leftrightarrow 3; \qquad 2 \to 2; \qquad 4 \leftrightarrow 7; \qquad 5 \leftrightarrow 6.$$

Thus $C(D) = (1, 3, 0)$ for the major-diatonic word D. This separates two claims that are easy to conflate: absence of limited-transposition symmetry does not mean absence of ordered reversal structure.

IX. Compositional Use

The classification is not only descriptive. A self-reflecting rotation can be used as a point of balance: forward and reflected forms preserve the same ordered interval pattern. A mirror pair gives two directed versions of the same material.

In this structural sense, a self-reflecting rotation can function as a resonant center because inversion returns the same ordered identity. A mirror pair can function as directed tension because inversion changes identity. These are formal relations inside the interval-word system. Whether they are heard as consonance, dissonance, color, release, or instability depends on register, spacing, rhythm, instrumentation, and context.

In register, these classes can shape pitch color. Repeating M3 across four octaves, for example, turns its nine-step word into a larger symmetrical field in $\mathbb{Z}_{48}$. The result is not a new acoustic timbre by itself; it is a voicing resource. Register, orchestration, and duration can turn that pitch structure into a distinct color.

The analogy is the fully diminished seventh chord: its symmetry supports several conventional resolutions. The present structures do not inherit those resolutions automatically, but they create comparable choice points. A composer can route a lifted word toward chromatic saturation, toward a diatonic collection, or toward another symmetric collection by choosing which tones continue and which resolve.

X. Multi-Octave Lifts

The one-octave model collapses register. A lift asks what survives when the same interval logic spans several octaves.

For height h, define the cyclic lift space $\mathbb{Z}_{12h}$. A one-octave word w with total span 12 may be repeated h times:

$$L_h^{\text{rep}}(w) = \underbrace{w\|w\|\cdots\|w}_{h \text{ times}},$$

with total span $12h$. Other lift operators use the reflection system rather than copying the same rotation. If u_i is a modal rotation, then

$$L_h^{\text{rot}}(u_i) = u_i\|u_{i+1}\|u_{i+2}\|\cdots,$$

where indices are read cyclically, gives a rotating lift. A mirror lift alternates a rotation with its reversed form:

$$L_h^{\text{mir}}(u_i) = u_i\|\mathsf{rev}(u_i)\|u_i\|\mathsf{rev}(u_i)\|\cdots.$$

A reflection-orbit lift follows the target map:

$$L_h^{\text{orb}}(u_i) = u_i\|u_{\mathsf{tgt}_w(i)}\|u_i\|u_{\mathsf{tgt}_w(i)}\|\cdots,$$

for a mirror pair, and collapses to repeated copies for a fixed point.

For M3, this means $u_1\|u_2\|u_1\|u_2$ creates an alternating mirror field, while $u_3\|u_3\|u_3\|u_3$ creates a vertically stable self-reflecting field. The finite one-octave table therefore controls phase from octave to octave.

In a radial drawing, pitch class determines angle and octave determines radius:

$$\theta(x) = 2\pi(x \bmod 12)/12, \qquad \rho(x) = 1 + \lfloor x/12 \rfloor.$$

In a helical drawing, pitch class determines angle and register determines height:

$$(\theta(x), z(x)) = (2\pi(x \bmod 12)/12, \lfloor x/12 \rfloor).$$

This construction differs from n-TET generalization [2] and from non-octave-repeating scale theory [7]. The question here is narrower: does the reflection relation survive when the same word is unfolded into register? The generator includes selected lift tables for heights 2, 3, and 4, but the one-octave classification remains the main result.

The same method can be applied outside Messiaen's seven modes. Diatonic interval words, other symmetrical divisions, and non-12 equal temperaments may yield different reflection classes. Future work should test repeated, rotate-lifted, and mirror-lifted words in $\mathbb{Z}_{24}$ and $\mathbb{Z}_{36}$, then repeat the computation in other moduli.

XI. Beyond Messiaen

The construction has natural tiers of generality. In any N-TET system, an interval word is a positive tuple whose steps sum to N, and the same rotation, reversal, target-map, and lift operations apply. More generally, any cyclic scale with ordered measurable steps can be treated as a word, whether the steps are equal-tempered units, cents, ratios, generators, or abstract scale degrees.

Scale combinations give another extension. Words can be concatenated, superposed as pitch collections and then re-sorted into a new cyclic word, interleaved between voices, or used as a product-like layout with one word governing horizontal motion and another governing register. Each combined object can then be classified again by the same reflection-system method. In that broader setting, Messiaen's seven words are not the endpoint of the theory; they are a compact, historically familiar test case for the finite machinery.

XII. Artifact and Reproducibility

The artifact has three parts. The Python generator at `papers/messiaen-radial/scripts/generate_datasets.py` emits the JSON data used by the companion page, including the seven Messiaen words, all positive interval words summing to 12 up to cyclic rotation, named comparison words, and selected lift operators. Regenerate the dataset with:

```
python3 papers/messiaen-radial/scripts/
generate_datasets.py
```

A second generator at `papers/messiaen-radial/scripts/generate_sheet_music.py` emits LilyPond sources, MusicXML files, staff SVG previews, pitch-graph SVGs, and PDF score books for the modal rotations and lift datasets. Regenerate the full notation bundle with:

```
python3 papers/messiaen-radial/scripts/
generate_sheet_music.py -scope all
```

The Lean artifact at `papers/messiaen-radial/formal` independently states the finite classification. It is pinned to Lean 4.31.0 by `lean-toolchain` and has no Mathlib dependency. Check it with:

```
cd papers/messiaen-radial/formal
lake build
```

The build checks the class-label, target-vector, and matrix theorems for the seven Messiaen words, along with the M3 starter theorem and the diatonic comparison. The Lean code currently treats the table as a finite computable theorem over explicit interval words; a more abstract development could later prove general facts about reversal actions on arbitrary cyclic words.

XIII. Limits

Reversing an interval word models one kind of reflection. It does not model every musical sense of the term. A performer may hear register, harmonic function, color, or voice-leading proximity more strongly than interval-word reversal.

The seven canonical Messiaen modes are a small set. Wider claims require generalized limited-transposition collections, truncations, and non-12 equal divisions.

Resolution claims also need musical examples. The diminished-seventh analogy names a use of symmetry, not an identical harmonic function. Multi-octave lifts add another choice: repeating a word, rotating it at octave boundaries, and alternating it with its reverse are different operators.

XIV. Conclusion

Messiaen's modes of limited transposition are usually classified by pitch-class repetition under transposition. Ordered interval words expose another layer: modal rotations can reflect into themselves or into each other. For the seven canonical modes, that relation yields the reflection systems in Tables 1–4. The same finite construction gives a diatonic comparison and a compact target for machine-checked proof. The larger compositional idea is that these finite maps can drive lift choices, mirror alternations, inversional pivots, and graph-like voice-leading paths. The Messiaen table is the verified seed; broader cyclic-scale and scale-combination systems are the natural next layer.

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